# ECE317 : Feedback and Control

## Lecture :
## Frequency response

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# Course roadmap

*Modeling*

- Laplace transform
- Transfer function
- Block Diagram
- Linearization
- Models for systems
  - electrical
  - mechanical
  - example system

*Analysis*

- Stability
  - Pole locations
  - Routh-Hurwitz
- Time response
  - Transient
  - Steady state (error)
- Frequency response
  - Bode plot

*Design*

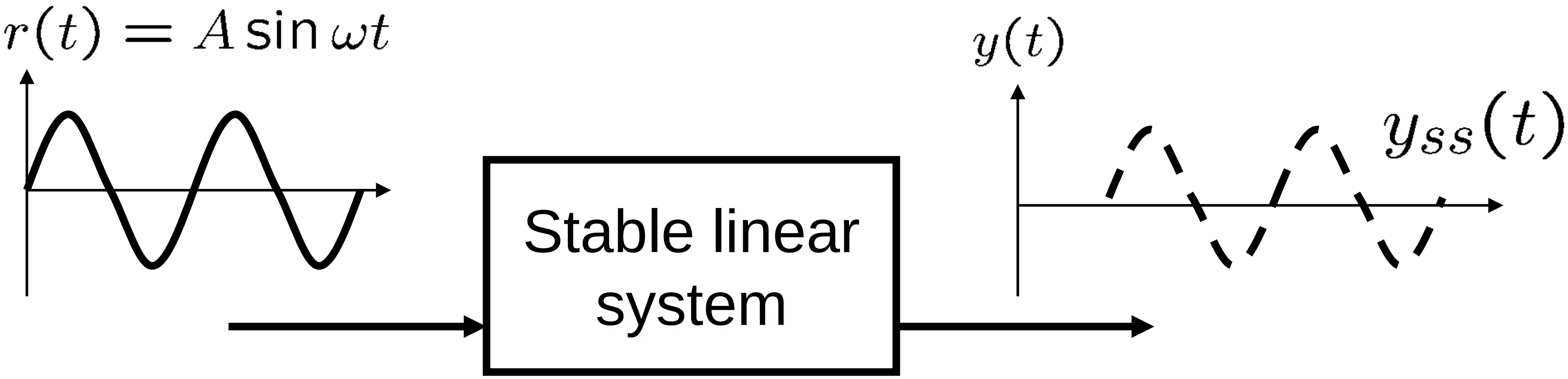
- Design specs
- Frequency domain
- Bode plot
- Compensation
- Design examples

*Matlab & PECS simulations & laboratories*

# Summary up to now
# & *Topics from now on*

- Modeling: How to represent systems with transfer functions (*s*-domain).
- Analysis: How to extract time-response information from *s*-function.
  - Steady-state error depends on TF evaluated at $s=0$.
    (to be covered later)
  - Stability and transient depends on pole locations.
  - *Frequency responses contain all these information.*
- Design: How to obtain "nice" closed-loop system.
  - *System's freq. responses can be shaped in Bode plot.*

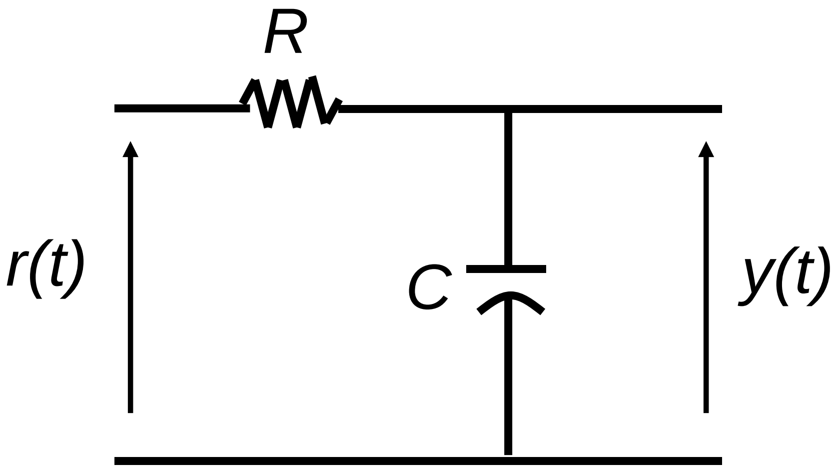
# What is frequency response?




- We would like to analyze a system property by applying a *sinusoidal input* *r(t)* and observing a response *y(t)*.
- Steady state response $y_{ss}(t)$ (after transient dies out) of a system to sinusoidal inputs is called ***frequency response***.

# A simple example

- RC circuit




$$G(s) = \frac{1}{RCs + 1}$$

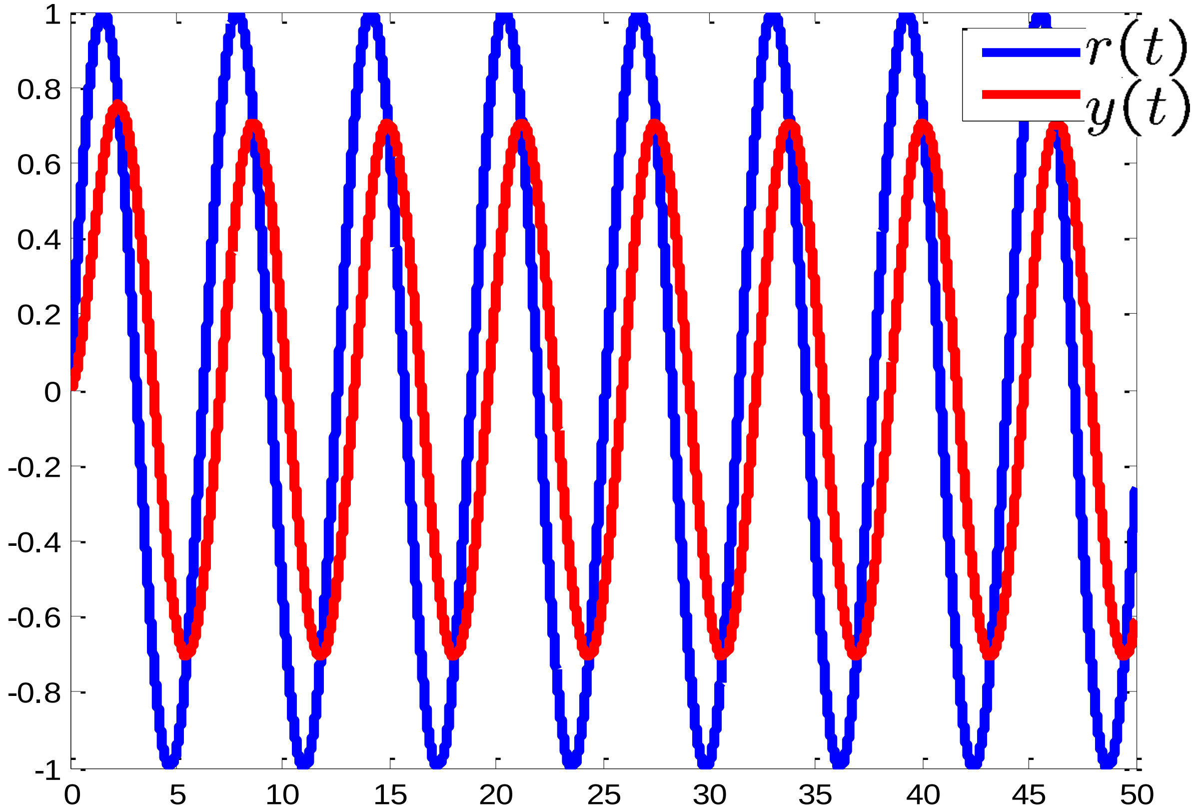
- Input a sinusoidal voltage *r(t)*
- What is the output voltage *y(t)*?

# An example (cont'd)

- TF (*R=C=1*)

$$G(s) = \frac{1}{s+1}$$

- *r(t)=sin(t)*




**At steady-state, *r(t)* and *y(t)* has same frequency, but different amplitude and phase!**

# An example (cont'd)

- Derivation of *y(t)*

$$Y(s) = G(s)R(s) = \frac{1}{s+1} \cdot \frac{1}{s^2+1} = \frac{1}{2}\left(\frac{1}{s+1} + \frac{-s+1}{s^2+1}\right)$$

Partial fraction expansion

- Inverse Laplace

$$y(t) = \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right)$$

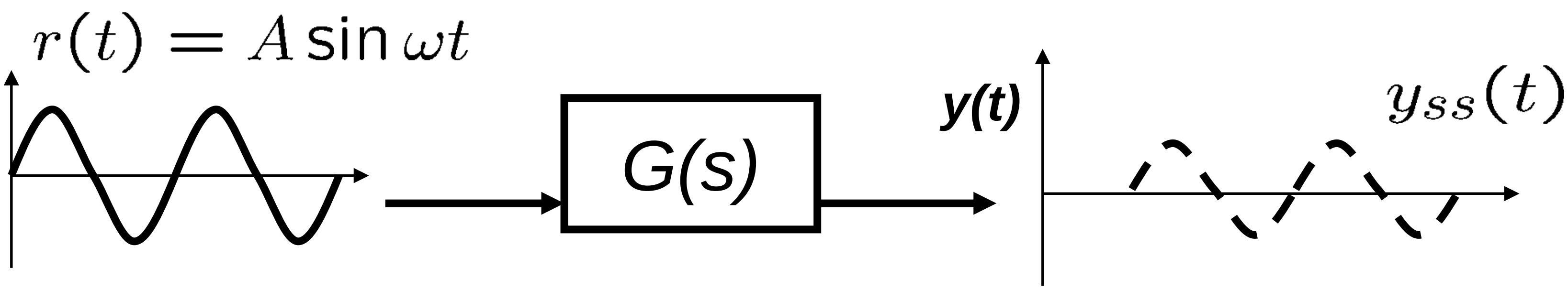
*0* as *t* goes to infinity.

$$\Rightarrow \quad y_{ss}(t) = \frac{1}{2}(-\cos t + \sin t) = \frac{1}{\sqrt{2}}\sin(t - 45^\circ)$$

(Derivation for general *G(s)* is given at the end of lecture slides.)

# Response to sinusoidal input

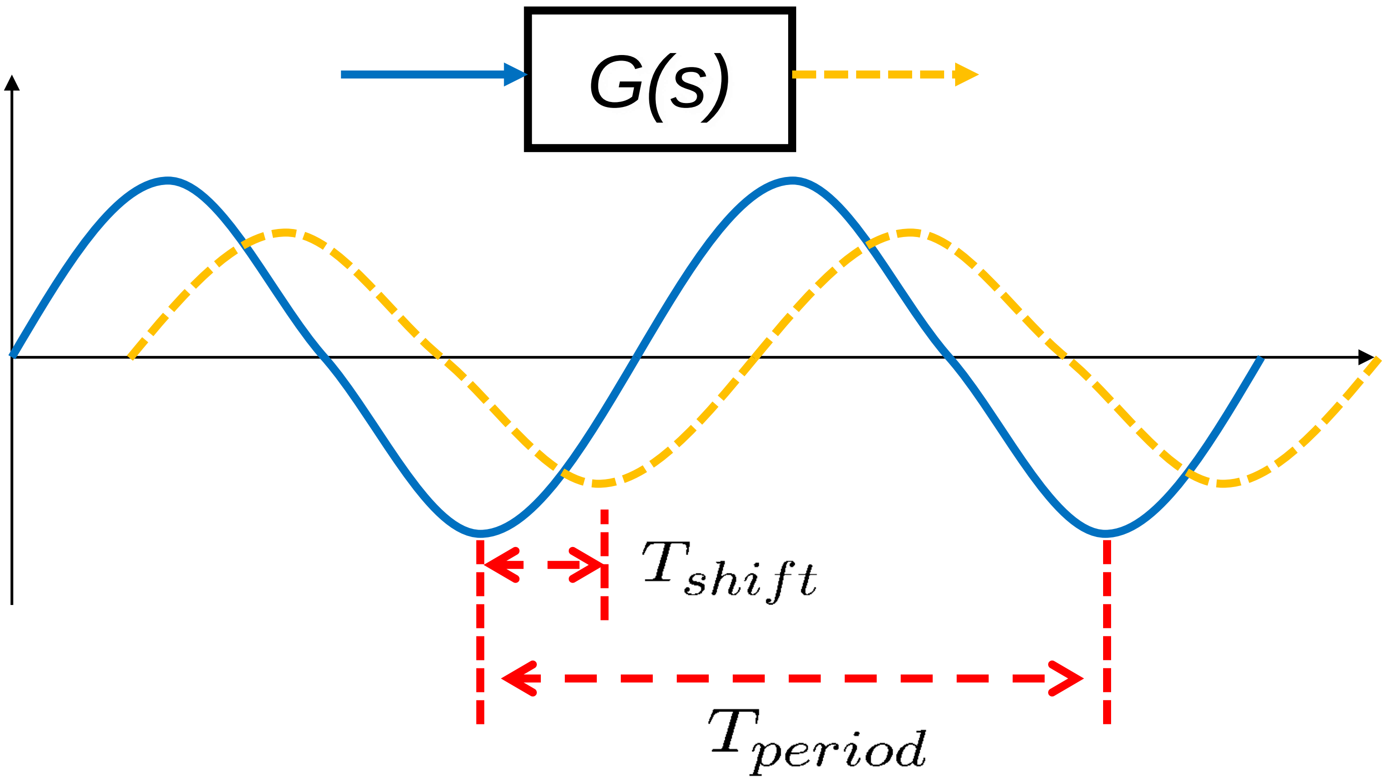
- What is the steady state output of a stable linear system when the input is sinusoidal?




- Steady state output $y_{ss}(t) = A\,|G(j\omega)|\sin(\omega t + \angle G(j\omega))$
  - Frequency is same as the input frequency $\omega$
  - Amplitude is that of input (A) multiplied by $|G(j\omega)|$
  - Phase shifts $\angle G(j\omega)$

Gain

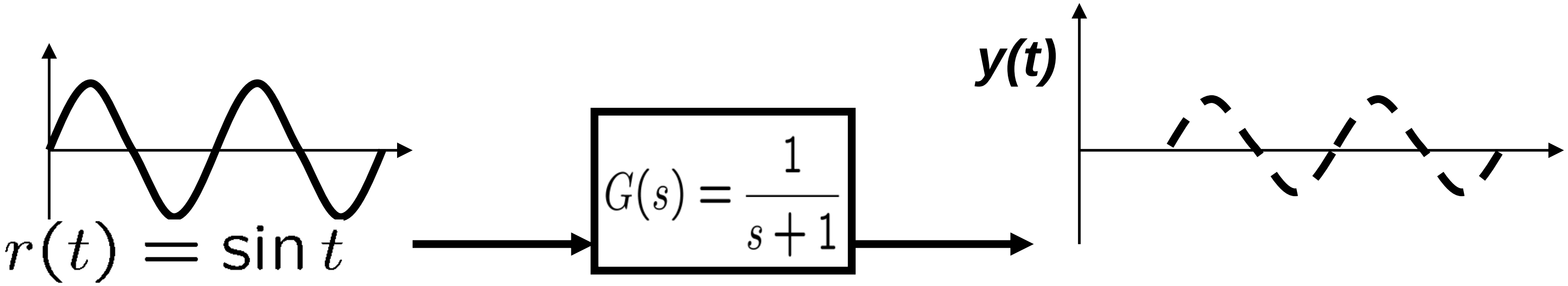
# Phase shift




$$\frac{T_{shift}}{T_{period}} = \frac{-\angle G(j\omega)}{360^\circ} \quad \Rightarrow \quad \angle G(j\omega) = -\frac{T_{shift}}{T_{period}} \times 360^\circ$$

# Revisit example

- How is the steady state output of a stable linear system when the input is sinusoidal?




$\omega = 1$

$$y_{ss}(t) = \underbrace{\frac{1}{\sqrt{2}}}_{|G(j\cdot 1)|} \sin(t \underbrace{-45^\circ}_{\angle G(j\cdot 1)})$$

Gain

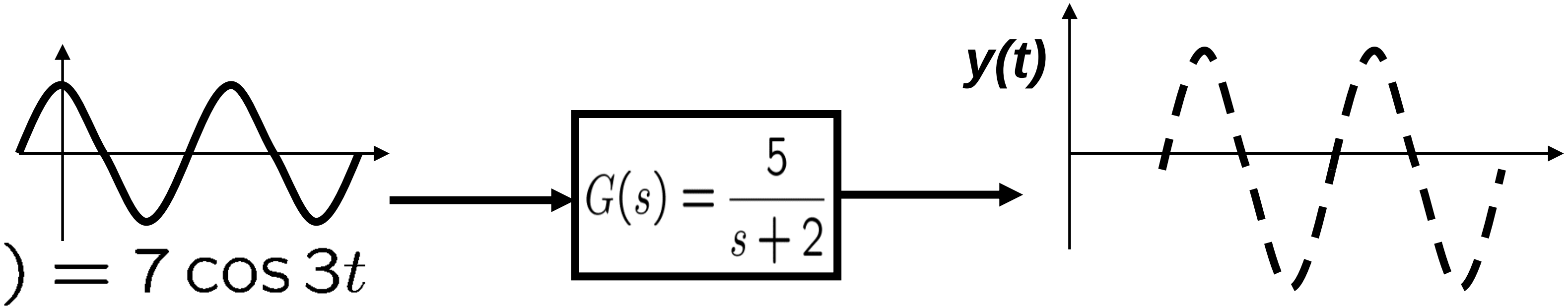
Phase

$$|G(j\cdot 1)| = \left|\frac{1}{j+1}\right| = \frac{1}{|j+1|} = \frac{1}{\sqrt{2}}$$

$$\angle G(j\cdot 1) = \angle \frac{1}{j+1} = \underbrace{\angle 1}_{=0} - \underbrace{\angle(j+1)}_{=45^\circ}$$

# Another example

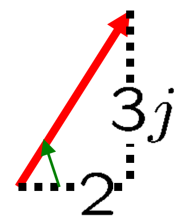
- How is the steady state output of a stable linear system when the input is sinusoidal?




$$r(t) = 7\cos 3t$$
$$\omega = 3$$

$$y_{ss}(t) = 7 \cdot \underbrace{\frac{5}{\sqrt{13}}}_{|G(j\cdot 3)|} \cos(3t + \underbrace{\theta}_{\angle G(j\cdot 3)})$$

Gain　　Phase



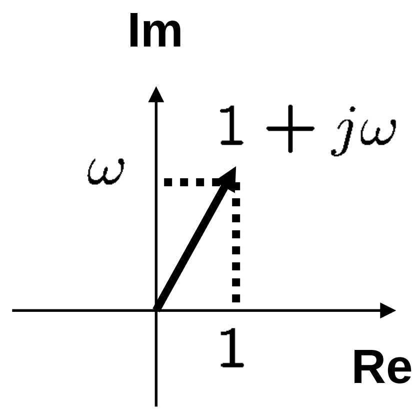

$$\angle G(j \cdot 3) = \angle \frac{5}{3j+2}$$
$$= \underbrace{\angle 5}_{=0} - \underbrace{\angle(3j+2)}_{=\tan^{-1}\frac{3}{2}}$$

# Frequency response function

- For a stable system *G(s)*, *G(jω)* (ω is positive) is called *frequency response function (FRF)*.
- For each ω, FRF takes a complex number *G(jω)*, which has a gain and a phase.
- First order example

$$G(s) = \frac{1}{s+1} \Rightarrow G(j\omega) = \frac{1}{j\omega + 1}$$

$$\Rightarrow \begin{cases} |G(j\omega)| = \dfrac{1}{\sqrt{1+\omega^2}} \\ \angle G(j\omega) = \angle(1) - \angle(j\omega + 1) = -\tan^{-1}\omega \end{cases}$$



# First order example (cont'd)

- FRF $G(j\omega) = \dfrac{1}{j\omega + 1}$

| frequency $\omega$ | gain $\lvert G(j\omega)\rvert$ | phase $\angle G(j\omega)$ |
|---|---|---|
| 0 | 1 | $0^\circ$ |
| 0.5 | 0.894 | $-26.6^\circ$ |
| 1.0 | 0.707 | $-45^\circ$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $\infty$ | 0 | $-90^\circ$ |

- Graph representing FRF
  - Bode diagram (Bode plot)

    (this course puts a lot of emphasis on Bode plot for analysis and design)

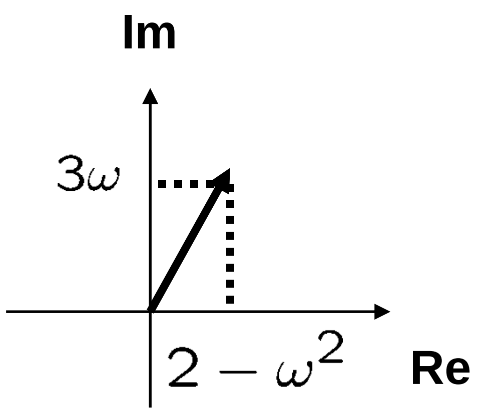
# Another example of FRF

- Second order system

$$G(s) = \frac{2}{s^2 + 3s + 2}$$

$$\Rightarrow \quad G(j\omega) = \frac{2}{(j\omega)^2 + 3(j\omega) + 2} = \frac{2}{2 - \omega^2 + j \cdot 3\omega}$$

$$\Rightarrow \quad \begin{cases} |G(j\omega)| = \dfrac{2}{\sqrt{(2-\omega^2)^2 + 9\omega^2}} \\ \angle G(j\omega) = \angle(2) - \angle(2 - \omega^2 + j \cdot 3\omega) \\ \qquad\qquad = -\tan^{-1} \dfrac{3\omega}{2 - \omega^2} \end{cases}$$



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- ✓ Time response
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  - Steady state (error)
- ➤ Frequency response
  - ➤ Bode plot

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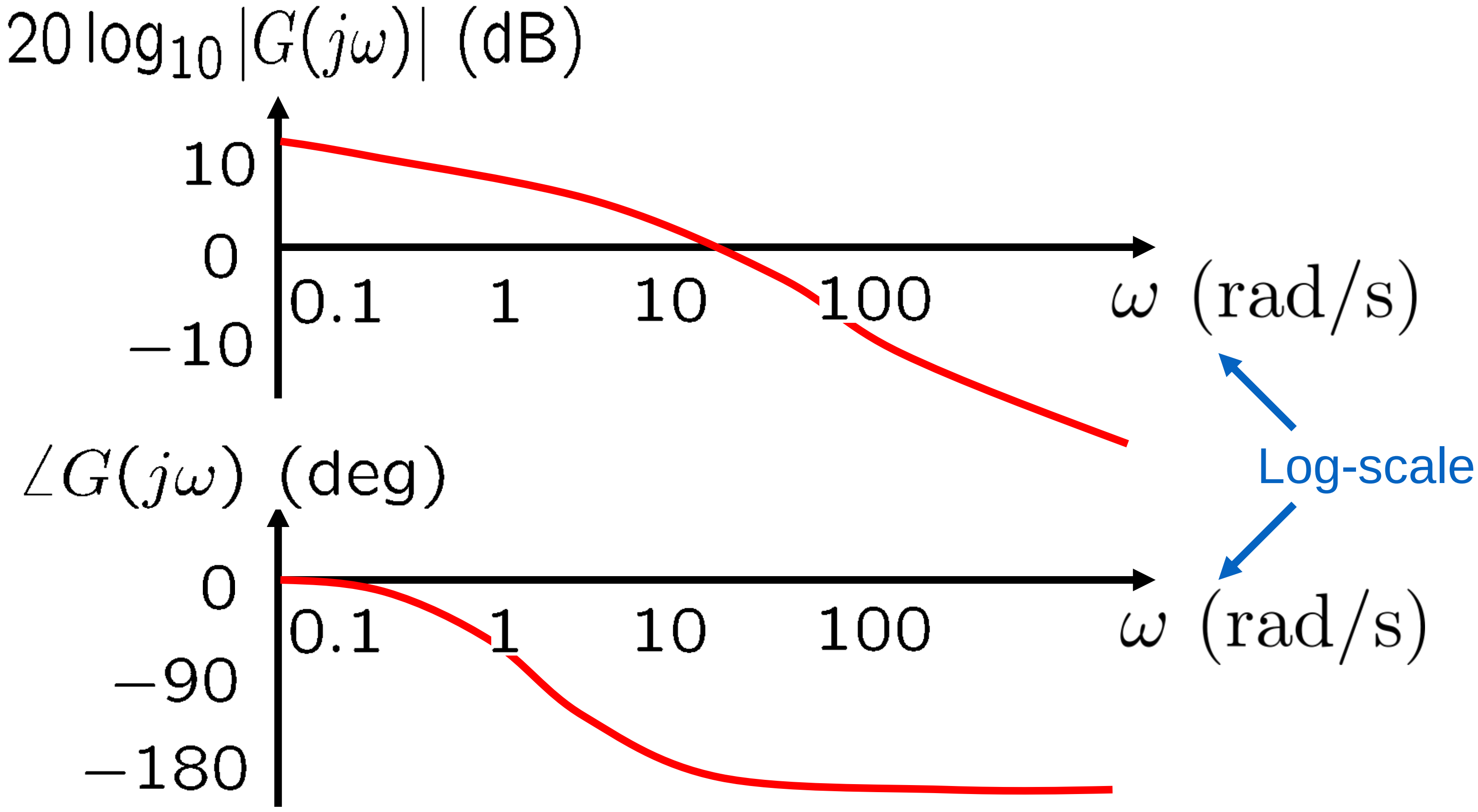
# Bode plot (Bode diagram) of *G(jω)*

- Bode diagram consists of gain plot & phase plot



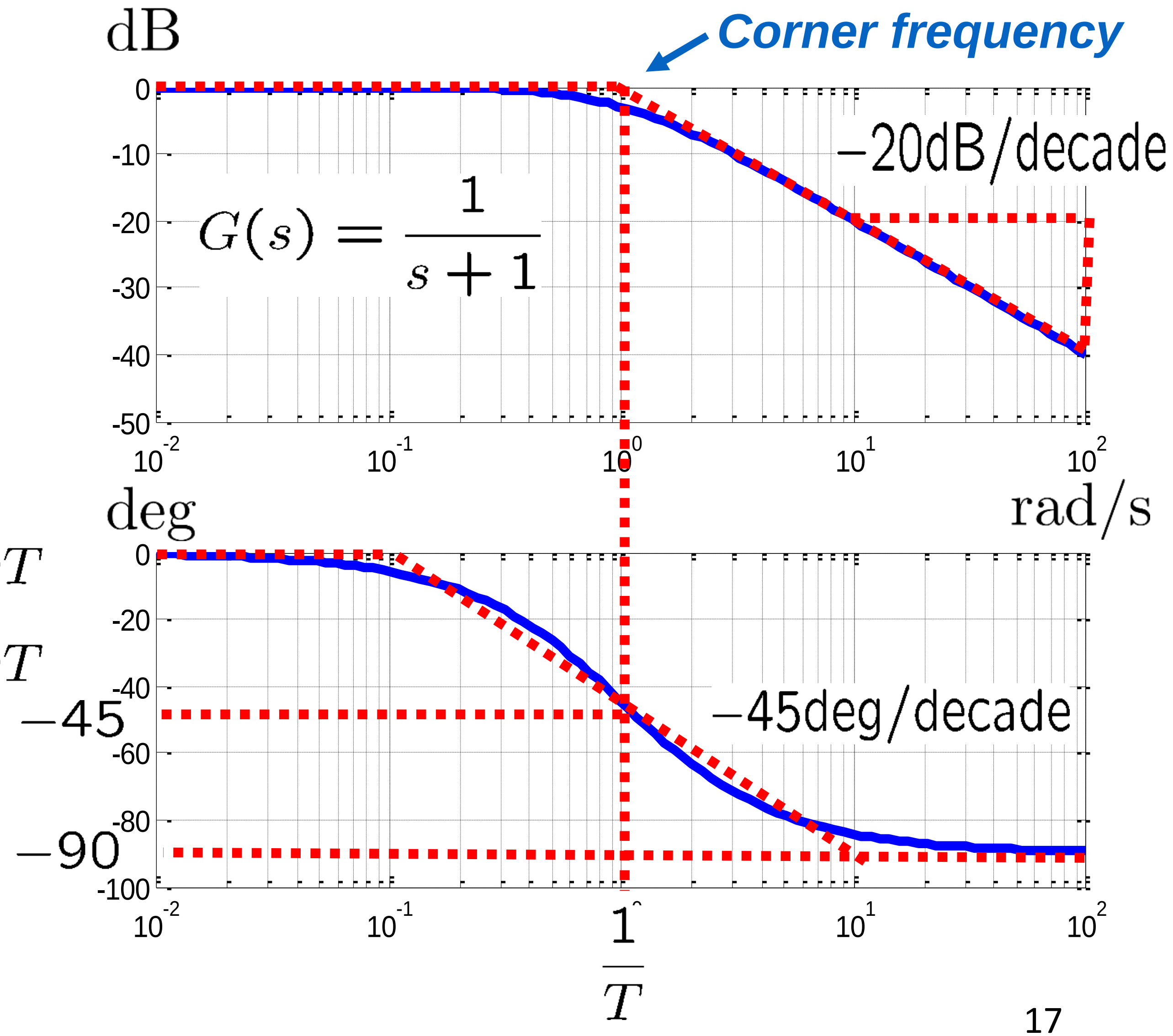
# Bode plot of a 1st order system

- TF

$$G(s) = \frac{1}{Ts+1}$$

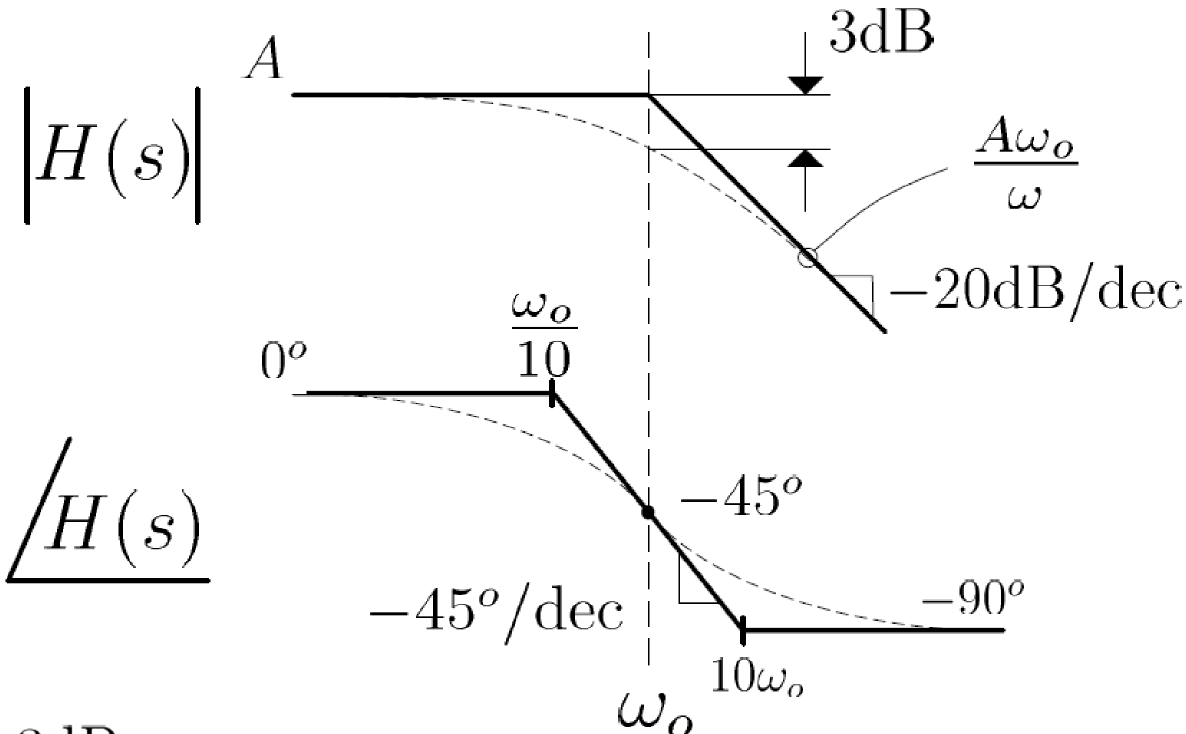
$$G(j\omega) = \frac{1}{j\omega T+1}$$

$$\approx \begin{cases} 1 & \text{if } 1 \gg \omega T \\ \frac{1}{j\omega T} & \text{if } 1 \ll \omega T \end{cases}$$



# Bode plot of a 1st order system, cont'd - determining values from annotations

$$H(s) = \frac{A}{1 + \frac{s}{\omega_o}}$$




Maximum Error @ $\omega_o = 3\text{dB}$

Maximum Error @ $\frac{\omega_o}{10}$ & $10\omega_o = 5.7^o$

Exact Phase: $-\tan^{-1}\left(\frac{\omega}{\omega_o}\right), \forall\omega$

# Exercises of sketching Bode plot

- First order systems

a) $$G(s) = \frac{1}{s+1}$$

b) $$G(s) = \frac{2}{0.1s+1}$$

c) $$G(s) = \frac{5}{10s+1}$$

# Remarks on Bode diagram

- Bode diagram shows gain and phase shift of a system output for sinusoidal inputs with various frequencies.
- Bode diagram is very useful and important in analysis and design of control systems.
- The shape of Bode plot contains information of CL stability, time responses, and much more!
- It can also be used for system identification. (Given FRF experimental data, obtain a transfer function that matches the data.)

# System identification

- Sweep frequencies of sinusoidal signals and obtain FRF data (i.e., gain and phase).
- Select *G(s)* so that $G(j\omega)$ fits the FRF data.

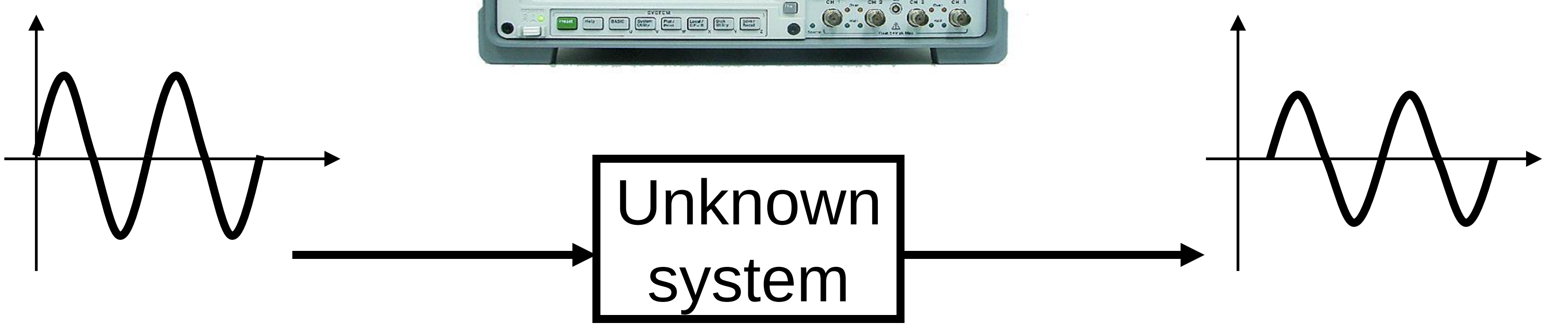
Agilent Technologies: FFT Dynamic Signal Analyzer

**Generate sin signals**

**Sweep frequencies**

**Collect FRF data**

**Select *G(s)***



# Summary

- Frequency response
  - Steady state response to a sinusoidal input
  - For a linear stable system, a sinusoidal input generates a sinusoidal output with same frequency but different amplitude and phase.
- Bode plot is a graphical representation of frequency response function. (MATLAB command “bode.m”)
- Next, how to sketch Bode plots

# Appendix
# Derivation of frequency response

$$Y(s) = G(s)R(s) = G(s)\frac{A\omega}{s^2+\omega^2} = \frac{k_1}{s+j\omega} + \frac{k_2}{s-j\omega} + C_g(s)$$

Term having denominator of stable *G(s)*

$$\begin{cases} k_1 = \lim_{s\to -j\omega}(s+j\omega)G(s)\frac{A\omega}{s^2+\omega^2} = G(-j\omega)\frac{A\omega}{-2j\omega} = -\frac{AG(-j\omega)}{2j} \\ k_2 = \lim_{s\to j\omega}(s-j\omega)G(s)\frac{A\omega}{s^2+\omega^2} = G(j\omega)\frac{A\omega}{2j\omega} = \frac{AG(j\omega)}{2j} \end{cases}$$

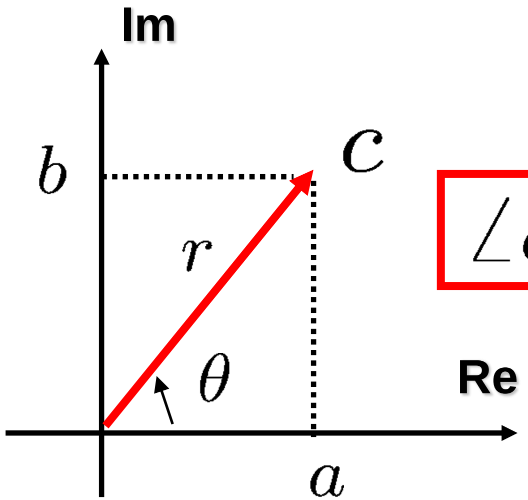
*0* as *t* goes to infinity.

$$\Rightarrow \quad y(t) = k_1 e^{-j\omega t} + k_2 e^{j\omega t} + \mathcal{L}^{-1}\{C_g(s)\}$$

$$\Rightarrow \quad y_{ss}(t) = A|G(j\omega)|\underbrace{\frac{e^{j(\omega t + \angle G(j\omega))} - e^{-j(\omega t + \angle G(j\omega))}}{2j}}_{\sin(\omega t + \angle G(j\omega))}$$

# Appendix
# Complex numbers (review)

- Representation
  - Cartesian form
    $$c = a + bj$$
  - Polar form
    $$c = re^{j\theta}$$




$$\angle c = \theta$$

- Multiplication & division in the polar form

$$\left.\begin{aligned} c_1 &= r_1 e^{j\theta_1} \\ c_2 &= r_2 e^{j\theta_2} \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 c_2 &= r_1 r_2 e^{j(\theta_1 + \theta_2)} \\ \frac{c_1}{c_2} &= \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)} \end{aligned}$$

# Appendix
# Why deg(den)>=deg(num)?

- All the transfer functions we encountered so far have the property $\deg(\mathrm{den}) \geq \deg(\mathrm{num})$

Ex: $\dfrac{1}{Ms^2 + Bs + K}$ $\dfrac{K}{Ts + 1}$ $K\dfrac{s+z}{s+p}$

- What if deg(num) is larger than deg(den)?
  - Then, $|G(j\omega)| \to \infty \text{ as } \omega \to \infty$
  - However, there is no such system in reality that has increasing gain as input frequency increases to infinity.
- That is why all the transfer function needs to meet $\deg(\mathrm{den}) \geq \deg(\mathrm{num})$